[Section-A] (Start to write From here)

Q.[1] (a) If $u = \tan^{-1}\left\{\frac{x^3+y^3}{x-y}\right\}$ Prove that (i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$

Sol. given $u = \tan^{-1}\left\{\frac{x^3+y^3}{x-y}\right\}$ Then $\tan u = f(u) = \frac{x^3+y^3}{x-y} = \frac{x^{\cancel{3}\,2}}{\cancel{x}}\left(\frac{1+\left(\frac{y}{x}\right)^3}{1-\frac{y}{x}}\right)$

$f(u) = \tan u = x^2\phi\left(\frac{y}{x}\right)$ Here $f(u) = \tan u$ is the homogeneous function of degree $n=2$ we have by Deduction of Euler's Theorem.

(i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = g(u) = \frac{nf(u)}{f'(u)} = \frac{2\tan u}{\sec^2 u} = \frac{2\sin u \cdot \cos^2 u}{\cos u}$

$= \sin 2u$ Pro

Q. 1 (b) If $\bar{q} = (y+z)\hat{i} + (x+z)\hat{j} + (x+y)\hat{k}$ then show that $\bar{q}$ is Irrotational.

Sol:- We have a vector field $\vec{F}$ is Irrotational vector field.

Then $\quad curl\,\vec{F} = 0$

i.e. $\quad \vec{\nabla}\times\vec{F} = 0$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & z+x & x+y \end{vmatrix}$$

$$curl\,\bar{q} = = \hat{i}\left(\frac{\partial}{\partial y}(x+y) - \frac{\partial}{\partial z}(z+x)\right) - \hat{j}\left(\frac{\partial}{\partial x}(x+y) - \frac{\partial}{\partial z}(y+z)\right)$$
$$+\hat{k}\left(\frac{\partial}{\partial x}(z+x) - \frac{\partial}{\partial y}(y+z)\right)$$

$$\vec{\nabla}\times\bar{q} = \hat{i}(1-1) - \hat{j}(1-1) + \hat{k}(1-1)$$

$$= 0\hat{i} - 0\hat{j} + 0\hat{k} = 0$$

Hence $\bar{q}$ is Irrotational vector field

Based

(c) Define Beta and Gamma function.

Beta function :-

it is denoted by $\beta(m,n)$ and define as

$$\boxed{\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1}\, dx} \quad m>0 \quad n>0$$

Gamma function :-

it is denoted by Γn

and define as $$\boxed{\Gamma n = \int_0^\infty e^{-x}\, x^{n-1}\, dx} \quad n>0$$

all the functions are the Type of Improper Integral.

Ans.

(d) Find the equation of Tangent plane and Normal line to the surface.

$$\phi = x^2+y^2+z-9=0 \quad \text{at} \quad P(1,2,4)$$

Sol:- given $\phi = x^2+y^2+z-9=0$

we have eqn of Tangent plane.

$$(x-x_0)f_x(P) + (y-y_0)f_y(P) + (z-z_0)f_z(P) = 0 \quad \text{—} (1)$$

and Normal line:-

$$\frac{x-x_0}{f_x(P)} = \frac{y-y_0}{f_y(P)} = \frac{z-z_0}{f_z(P)} = t \quad \text{—} (2)$$

Partial diff w.r.to x y and z.

$$\frac{\partial\phi}{\partial x} = 2x, \quad \frac{\partial\phi}{\partial y} = 2y \quad \frac{\partial\phi}{\partial z} = 1$$

$$\phi_x(P) = 2 \quad \phi_y(P) = 4 \quad \phi_z(P) = 1$$

Then Normal line:-

$$\boxed{\frac{x-1}{2} = \frac{y-2}{4} = \frac{z-4}{1}}$$

Tangent plane:-

$$(x-1)\cdot 2 + (y-2)\cdot 4 + (z-4)1 = 0$$

$$\boxed{2x+4y+z=14} \quad \text{Ans.}$$

Q 1 (e) Find the stationary point of the following function

$$u(x,y) = x^2+y^2+\frac{2}{x}+\frac{2}{y} \quad \text{--- (1)}$$

Sol:- given

$$u(x,y) = x^2+y^2+\frac{2}{x}+\frac{2}{y}$$

(I) Find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$

$\frac{\partial u}{\partial x} = 2x-\frac{2}{x^2}$ and $\frac{\partial u}{\partial y} = 2y-\frac{2}{y^2}$

(II) equating $\frac{\partial u}{\partial x}=0$ and $\frac{\partial u}{\partial y}=0$

Then $2x-\frac{2}{x^2}=0$ --- (1) and $2y-\frac{2}{y^2}=0$ --- (2)

$x^3-1=0$ $\qquad$ $y^3-1=0$

$(x-1)=0$ $\qquad$ $(y-1)=0$

Then $x=1$ $y=1$

ie The stationary point is $(1,1)$ at which we check the Maxima and Minima of the function

Ans.

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[Part - B]

Q. 2 a If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$ Then show That

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$$

Sol :- given $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$ let $t_1 = \frac{x}{y}$

$u = f(t_1, t_2, t_3)$ and t_1, t_2, t_3 are the function of x y and z. $t_2 = \frac{y}{z}$

$t_3 = \frac{z}{x}$

Now $\frac{\partial t_1}{\partial x} = \frac{1}{y}$ $\frac{\partial t_1}{\partial y} = -\frac{x}{y^2}$ $\frac{\partial t_1}{\partial z} = 0$

$\frac{\partial t_2}{\partial x} = 0$ $\frac{\partial t_2}{\partial y} = \frac{1}{z}$, $\frac{\partial t_2}{\partial z} = \frac{-y}{z^2}$

$\frac{\partial t_3}{\partial x} = \frac{-z}{x^2}$ $\frac{\partial t_3}{\partial y} = 0$, $\frac{\partial t_3}{\partial z} = \frac{1}{x}$

We have by Total derivative.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t_1}\cdot\frac{\partial t_1}{\partial x} + \frac{\partial u}{\partial t_2}\cdot\frac{\partial t_2}{\partial x} + \frac{\partial u}{\partial t_3}\cdot\frac{\partial t_3}{\partial x}$$

$$= \frac{\partial u}{\partial t_1}\cdot\left(\frac{1}{y}\right) + \frac{\partial u}{\partial t_2}\times 0 + \frac{\partial u}{\partial t_3}\times\left(\frac{-z}{x^2}\right) \quad \text{— (1)} \times x$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial t_1}\cdot\frac{\partial t_1}{\partial y} + \frac{\partial u}{\partial t_2}\cdot\frac{\partial t_2}{\partial y} + \frac{\partial u}{\partial t_3}\cdot\frac{\partial t_3}{\partial y}$$

$$= \frac{\partial u}{\partial t_1}\left(\frac{-x}{y^2}\right) + \frac{\partial u}{\partial t_2}\left(\frac{1}{z}\right) + \frac{\partial u}{\partial t_3}\times 0 \quad \text{— (2)} \times y$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial t_1}\cdot\frac{\partial t_1}{\partial z} + \frac{\partial u}{\partial t_2}\cdot\frac{\partial t_2}{\partial z} + \frac{\partial u}{\partial t_3}\cdot\frac{\partial t_3}{\partial z}$$

$$= \frac{\partial u}{\partial t_1}\times 0 + \frac{\partial u}{\partial t_2}\times\frac{-y}{z^2} + \frac{\partial u}{\partial t_3}\times\frac{1}{x} \quad \text{— (3)} \times z.$$

adding (1) (2) and (3)

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \cancel{\frac{x}{y}\frac{\partial u}{\partial t_1}} - \cancel{\frac{z}{x}\frac{\partial u}{\partial t_3}} + \cancel{\left(\frac{-x}{y}\right)\frac{\partial u}{\partial t_1}} + \cancel{\frac{y}{z}\frac{\partial u}{\partial t_2}}$$

$$+ \cancel{\left(\frac{-y}{z}\right)\frac{\partial u}{\partial t_2}} + \cancel{\left(\frac{z}{x}\right)\frac{\partial u}{\partial t_3}} = 0$$

Hence $\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0}$ proved

A

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Q. 2 b Prove that $\beta(m,n) = \dfrac{\Gamma m \cdot \Gamma n}{\Gamma m+n}$ $m, n > 0$

Sol:- we have by property of Gamma function

$$\frac{\Gamma n}{z^n} = \int_0^\infty e^{-zx} \cdot x^{n-1}\, dx \quad \text{—— (i)}$$

Then

$$\Gamma n = \int_0^\infty e^{-zx} z^n x^{n-1}\, dx$$

Multiplying $e^{-z} z^{m-1}$ both side and Integrate with respect to z upto limit $z = 0$ to $z = \infty$

$$\Gamma n \int_0^\infty e^{-z} z^{m-1}\, dz = \int_0^\infty \int_0^\infty e^{-zx} z^n x^{n-1}\, dx \cdot e^{-z} z^{m-1}\, dz$$

$$\Gamma n \cdot \Gamma m = \int_0^\infty e^{-(1+x)z} \cdot z^{m+n-1}\, dz \cdot \int_0^\infty x^{n-1}\, dx$$

$$\Gamma n \cdot \Gamma m = \int_0^\infty e^{-(1+x)z} \cdot z^{m+n-1}\, dx \cdot \int_0^\infty x^{n-1}\, dx$$

$$\Gamma n \cdot \Gamma m = \frac{\Gamma m+n}{(1+x)^{m+n}} \int_0^\infty x^{n-1}\, dx \qquad \because \frac{\Gamma n}{z^n} = \int_0^\infty e^{-zx} \cdot x^{n-1}\, dx$$

$$\Gamma n \cdot \Gamma m = \Gamma m+n \int_0^\infty \frac{x^{n-1}\, dx}{(1+x)^{m+n}}.$$

$$\Gamma n\, \Gamma m = \Gamma m+n \cdot \beta(m, n) \qquad \because \beta(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\, dx$$

$$\therefore \boxed{\beta(m,n) = \frac{\Gamma n \cdot \Gamma m}{\Gamma m+n}}$$

Proved

Ans.

Q ② ⓒ Find the divergence and curl of the vector

$$\vec{F} = \nabla(x^3+y^3+z^3-3xyz)$$

Sol:- given $\vec{F} = \nabla(x^3+y^3+z^3-3xyz)$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(x^3+y^3+z^3-3xyz)$$

$$\vec{F} = (3x^2-3yz)\hat{i} + (3y^2-3zx)\hat{j} + (3z^2-3xy)\hat{k}$$

Then $\text{Div}\,\vec{F} = \vec{\nabla}\cdot\vec{F} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\vec{F}$ $\qquad \hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

$$= \frac{\partial}{\partial x}(3x^2-3yz) + \frac{\partial}{\partial y}(3y^2-3zx) + \frac{\partial}{\partial z}(3z^2-3xy)$$

$$= 6x + 6y + 6z.$$

$$= 6(x+y+z)$$

Ans.

Now $\text{curl}\,\vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \end{vmatrix}$

$$= \hat{i}\left(\frac{\partial}{\partial y}(3z^2-3xy) - \frac{\partial}{\partial z}(3y^2-3xz)\right) - \hat{j}\left(\frac{\partial}{\partial x}(3z^2-3xy) - \frac{\partial}{\partial z}(3x^2-3yz)\right)$$
$$+ \hat{k}\left(\frac{\partial}{\partial x}(3y^2-3xz) - \frac{\partial}{\partial y}(3x^2-3yz)\right)$$

$$\text{curl}\,\vec{F} = \vec{\nabla} \times \vec{F} = \hat{i}(\cancel{-3x}+\cancel{3x}) - \hat{j}(\cancel{-3y}+\cancel{3y}) + \hat{k}(\cancel{-3z}+\cancel{3z})$$

$$= 0\hat{i} - 0\hat{j} + 0\hat{k}$$
$$= 0$$

Hence $\text{curl}\,\vec{F} = 0$ so that $\vec{F}$ is Irrotational vector field.

Ans.

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(d) If $u = f(r)$ where $r^2 = x^2 + y^2$

Prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r)$$

Sol:- given $u = f(r)$, $r^2 = x^2 + y^2$

$\left\{ \frac{\partial r}{\partial x} = \frac{x}{r} , \frac{\partial r}{\partial y} = \frac{y}{r} \right.$

Then Partial Diff w.r.to x

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} = \frac{du}{dr} \cdot \frac{\partial r}{\partial x} = f'(r) \cdot \frac{x}{r} \quad \text{——} (1)$$

again P. Diff. w.r.to x

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(f'(r) \cdot \frac{x}{r} \right)$$

$$= f'(r) \left(\frac{r \cdot \cancel{1} - x \cdot \frac{\partial r}{\partial x}}{r^2} \right) + \frac{x}{r} \cdot f''(r) \cdot \frac{\partial r}{\partial x}$$

$$= f'(r) \left(\frac{r - x \cdot \frac{x}{r}}{r^2} \right) + \frac{x^2}{r^2} f''(r)$$

$$\frac{\partial^2 u}{\partial x^2} = f'(r) \left[\frac{r^2 - x^2}{r^3} \right] + \frac{x^2}{r^2} f''(r) \quad \text{——} (2)$$

Similarly $\frac{\partial^2 u}{\partial y^2} = f'(r)\left[\frac{r^2-y^2}{r^3}\right] + \frac{y^2}{r^2} f''(r)$ —— (3)

adding (2) and (3)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f'(r)\left[\frac{2r^2-(x^2+y^2)}{r^3}\right] + \frac{f''(r)}{r^2}(x^2+y^2)$$

$\because x^2+y^2=r^2$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f'(r)\left[\frac{2r^2-r^2}{r^3}\right] + \frac{f''(r)}{\cancel{r^2}}\cdot\cancel{r^2}$$

$$= f'(r)\left[\frac{\cancel{r^2}}{r\cancel{^3}}\right] + f''(r)$$

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r)}$$

Proved

Ans.

(e) Find The extreme value of $x^2+y^2+z^2$ subject to The condition $ax+by+cz=p$

Sol:- given $f(x,y,z) = x^2+y^2+z^2$

and

Subject to $\phi = ax+by+cz-p=0$

we have by Lagrange's Multiplier Method.

we have by Lagrange's eqn.

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0 \quad \text{--- (1)}$$

$$\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0 \quad \text{--- (2)}$$

$$\frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0 \quad \text{--- (3)}$$

where λ is The Lagrange's Multiplier.

Now $\frac{\partial f}{\partial x} = 2x$, $\frac{\partial f}{\partial y} = 2y$ $\frac{\partial f}{\partial z} = 2z$.

and $\frac{\partial \phi}{\partial x} = a$ $\frac{\partial \phi}{\partial y} = b$, $\frac{\partial \phi}{\partial z} = c$

Then $2x + a\lambda = 0$ — (1)

$2y + b\lambda = 0$ — (2)

$2z + c\lambda = 0$ — (3)

$x = \frac{-a\lambda}{2}$ $y = \frac{-b\lambda}{2}$ $z = \frac{-c\lambda}{2}$ Putting these value in $ax + by + cz = p$

Then

$$-\frac{a^2}{2}\lambda - \frac{b^2}{2}\lambda - \frac{c^2}{2}\lambda = p$$

$$\boxed{\lambda = \frac{-2p}{a^2+b^2+c^2}}$$

Then

$$x = -\frac{a}{\cancel{2}} \times \frac{-\cancel{2}p}{a^2+b^2+c^2} = +\frac{ap}{a^2+b^2+c^2}$$

$$y = \frac{bp}{a^2+b^2+c^2}$$

$$z = \frac{cp}{a^2+b^2+c^2}$$

$$\boxed{x = \frac{ap}{a^2+b^2+c^2}} \quad \boxed{y = \frac{bp}{a^2+b^2+c^2}} \quad \boxed{z = \frac{cp}{a^2+b^2+c^2}}$$

These are the extreme value of

$$f(x,y,z) = x^2+y^2+z^2$$

$$= \frac{a^2p^2}{(a^2+b^2+c^2)^2} + \frac{b^2p^2}{(a^2+b^2+c^2)^2} + \frac{c^2p^2}{(a^2+b^2+c^2)^2} = \frac{\cancel{(a^2+b^2+c^2)}p^2}{(a^2+b^2+c^2)^{\cancel{2}}} = \frac{p^2}{(a^2+b^2+c^2)}$$

Ans

Q. (2) (f) If $f(x,y) = \begin{cases} \dfrac{x^3-y^3}{x^2+y^2} & \text{when } x \neq 0\ y \neq 0 \\ 0 & \text{when } x = y = 0 \end{cases}$

Then discuss the continuity of $f(x,y)$ at the origin.

Sol:- a function $f(x,y)$ is continuous at origin.

If

$$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) \quad \text{—— (1)}$$

Then.

Path – I.
$$\lim_{\substack{x\to 0 \\ y\to 0}} \left(\frac{x^3-y^3}{x^2+y^2}\right) = \lim_{y\to 0} \left\{\frac{-y^{\cancel{3}}}{\cancel{y^2}}\right\} = 0$$

Path – II
$$\lim_{\substack{y\to 0 \\ x\to 0}} \left(\frac{x^3-y^3}{x^2+y^2}\right) = \lim_{x\to 0} \left\{\frac{x^{\cancel{3}}}{\cancel{x^2}}\right\} = \lim_{x\to 0}\{x\} = 0$$

along the path $y = mx$ and $y = mx^2$

$$\lim_{\substack{y\to mx \\ x\to 0}} \left(\frac{x^3-y^3}{x^2+y^2}\right) = \lim_{x\to 0} \left\{\frac{x^3-m^3x^3}{x^2+m^2x^2}\right\} = \lim_{x\to 0} \left\{\frac{x^{\cancel{3}}(1-m^3)}{\cancel{x^2}(1+m^2)}\right\} = 0$$

$$\lim_{\substack{y \to mx^2 \\ x \to 0}} \left[\frac{x^3-y^3}{x^2+y^2}\right] = \lim_{x \to 0} \left[\frac{x^3-m^3x^6}{x^2+m^2x^4}\right] = \lim_{x \to 0} \left[\frac{\cancel{x^3}}{\cancel{x^2}} \frac{(1-m^3x^3)}{(1+m^2x^2)}\right] = 0$$

Now limit of $f(x,y)$ along all the Path is same and which is equal to the value of the function at origin. So The function is continuous.

ie $$\boxed{\lim_{(x,y) \to (0,0)} f(x,y) = 0 = f(0,0)}$$

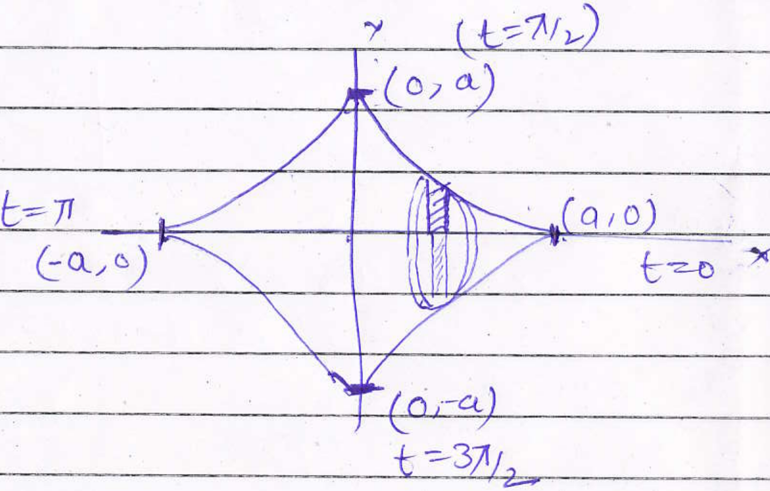
Ans.

[Part - C]

Q. (3) (a) Find the volume and surface area of solid generated by Revolving the curve ~~circle~~ astroid
$x = a\cos^3 t$, $y = a\sin^3 t$ about x-axis.

Sol:- given eqn of curve. (Astroid)
$x = a\cos^3 t$, $y = a\sin^3 t$ which is in Parameteric ~~function~~ form, t as a Parameter.

$\frac{dx}{dt} = 3a\cos^2 t \cdot (-\sin t)$



$\frac{dy}{dt} = 3a\sin^2 t \cdot \cos t$

we have.

surface.

$$S = \int_a^b 2\pi y \, ds$$

$$= \int_a^b 2\pi y \frac{ds}{dt} \cdot dt \qquad \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

Required surface $S = 2 \times \int_0^{\pi/2} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$

$$S = 4\pi \int_0^{\pi/2} a\sin^3 t \cdot \sqrt{9a^2 \cos^4 t \cdot \sin^2 t + 9a^2 \sin^4 t \cdot \cos^2 t}\; dt$$

$$S = 4\pi \int_0^{\pi/2} 3a^2 \sin^3 t \cdot \sin t \cos t \cdot \sqrt{\cos^2 t + \sin^2 t}\; dt$$

$$= 12\pi a^2 \int_0^{\pi/2} \sin^4 t \cdot \cos t \, dt$$

$\sin t = u$

$\cos t \, dt = du$

$$S = 12\pi a^2 \int_0^1 u^4 \, du = 12\pi a^2 \left[\frac{u^5}{5}\right]_0^1$$

$$\boxed{S = \frac{12\pi a^2}{5}}$$ Ans

Volume

$$V = 2 \times \int_0^{\pi/2} \pi y^2 \frac{dx}{dt} dt = 2 \times \int_0^{\pi/2} \pi (a\sin^3 t)^2 \cdot (-3a) \cos^2 t \sin t \, dt$$

$$V = -6\pi a^3 \int_0^{\pi/2} \sin^6 t \cos^2 t \, dt \;\; \sin t = -6\pi a^3 \int_0^{\pi/2} \sin^7 t \cos^2 t \, dt$$

$p = 7 \quad q = 2$

$$= -\cancel{6}^{3}\pi a^3 \times \frac{\Gamma\frac{7+1}{2}\,\Gamma\frac{2+1}{2}}{\cancel{2}\,\Gamma\frac{7+2+2}{2}} = \frac{-3\pi a^3 \times \lfloor 3 \times \Gamma\frac{3}{2}}{\Gamma\frac{11}{2}} =$$

$$V = \frac{-3\pi a^3 \times \lfloor 3 \times \lfloor \frac{3}{2}}{\frac{9}{2}\times\frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\,\lfloor\frac{3}{2}} \Rightarrow \frac{-\cancel{3}\pi a^3 \times \cancel{6}^{2} \times 16}{\underset{3}{\cancel{9}} \times 7 \times 5 \times \cancel{3}}$$

$$V = \frac{-32\pi a^3}{105} \quad \text{Ans.}$$

(Neglecting -ve sign)

$$\boxed{V = \frac{32\pi a^3}{105}}$$

Q. 3 b (I) Evaluate

$$I = \int_0^\infty \frac{dx}{1+x4}$$

let $x^2 = \tan\theta$ $\qquad x = \sqrt{\tan\theta}$

$2x\,dx = \sec^2\theta\, d\theta$

$$= \int_0^{\pi/2} \frac{1}{1+\tan^2\theta} \cdot \frac{\sec^2\theta\, d\theta}{2\sqrt{\tan\theta}}$$

$$\frac{1}{2}\int_0^{\pi/2} \frac{1}{\cancel{\sec^2\theta}} \cdot \frac{\cancel{\sec^2\theta}}{\sqrt{\tan\theta}}\, d\theta = \frac{1}{2}\int_0^{\pi/2} \frac{1}{\sqrt{\tan\theta}}\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} (\tan\theta)^{-1/2}\, d\theta = \frac{1}{2}\int_0^{\pi/2} (\cot\theta)^{1/2}\, dt = \frac{1}{2}\int_0^{\pi/2} \cos^{1/2}\theta \sin^{-1/2}\theta\, d\theta$$

$q = 1/2 \quad p = -1/2$

we have $\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta$

$$= \frac{\Gamma\left(\frac{p+1}{2}\right) \cdot \Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{p+q+2}{2}\right)}$$

$\int_0^{\pi/2} \cos^{1/2}\theta \cdot \sin^{-1/2}\theta \, d\theta$ $\quad p = -1/2 \quad q = 1/2$

$$= \frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right) \Gamma\left(\frac{\frac{1}{2}+1}{2}\right)}{2\,\Gamma\left(\frac{-\frac{1}{2}+\frac{1}{2}+2}{2}\right)} = \frac{\Gamma\frac{1}{4}\,\Gamma\frac{3}{4}}{2\,\Gamma 1} \qquad \because \Gamma 1 = 1$$

$$= \frac{1}{2}\Gamma\frac{1}{4}\Gamma\frac{3}{4} \qquad \Gamma n\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}$$

$$= \frac{1}{2}\Gamma\frac{1}{4}\Gamma\left(1-\frac{1}{4}\right) = \frac{1}{2} \times \frac{\pi}{\sin \pi/4} = \frac{1}{2} \times \frac{\pi}{\frac{1}{\sqrt{2}}}$$

$$\frac{\pi}{2 \times \frac{1}{\sqrt{2}}} = \frac{\pi}{\sqrt{2}}$$ Ans.

Evaluate

(ii) $I = \int_0^{\pi/6} \cos^4 3\theta \cdot \sin^2 6\theta \, d\theta$ let $3\theta = t$

$d\theta = \frac{dt}{3}$

$\int_0^{\pi/2} \cos^4 t \cdot \sin^2 2t \, \frac{dt}{3}$

$= \frac{1}{3} \int_0^{\pi/2} \cos^4 t \cdot (2\sin t \cdot \cos t)^2 \, dt$

$= \frac{4}{3} \int_0^{\pi/2} \cos^6 t \sin^2 t \, dt$

$P = 2 \quad q = 6$

$= \frac{4}{3} \frac{\Gamma\frac{2+1}{2} \Gamma\frac{6+1}{2}}{2\Gamma\frac{2+6+2}{2}} = \frac{\cancel{4}^{2}}{3} \times \frac{\Gamma\frac{3}{2}\Gamma\frac{7}{2}}{\cancel{2}\Gamma 5}$

$= \frac{2}{3} \times \Gamma\frac{3}{2} \Gamma\frac{7}{2} \times \frac{1}{\Gamma 4}$

$= \frac{\cancel{2}}{\cancel{3}} \times \frac{1}{\cancel{2}} \times \sqrt{\pi} \times \frac{5}{2} \times \frac{\cancel{3}}{2} \times \frac{1}{2} \sqrt{\pi} \times \frac{1}{24} = \frac{5\pi}{24 \times 8}$

$= \frac{5\pi}{192}$

Ans.

Q. 3 c) If $u = f(x, y)$ where $x = \varphi \cos\alpha - \eta \sin\alpha$

$y = \varphi \sin\alpha + \eta \cos\alpha$

Then Prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial \eta^2}$$

Sol:- $u = f(x, y)$ $\quad x = \varphi \cos\alpha - \eta \sin\alpha$

$y = \varphi \sin\alpha + \eta \cos\alpha$

$$\left\{ \begin{aligned} \frac{\partial x}{\partial \varphi} &= \cos\alpha \\ \frac{\partial x}{\partial \eta} &= -\sin\alpha \\ \frac{\partial y}{\partial \varphi} &= \sin\alpha \\ \frac{\partial y}{\partial \eta} &= \cos\alpha \end{aligned} \right.$$

we have.

$$\frac{\partial u}{\partial \varphi} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \varphi} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \varphi}$$

$$\frac{\partial u}{\partial \varphi} = \frac{\partial u}{\partial x} \cdot \cos\alpha + \frac{\partial u}{\partial y} \cdot \sin\alpha \quad \text{--- (1)}$$

$$\frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \eta} = \frac{\partial u}{\partial x} \cdot (-\sin\alpha) + \frac{\partial u}{\partial y} \cdot \cos\alpha$$

$$= -\sin\alpha \frac{\partial u}{\partial x} + \cos\alpha \frac{\partial u}{\partial y} \quad \text{--- (2)}$$

$$\frac{\partial^2 u}{\partial \varphi^2} = \frac{\partial}{\partial \varphi}\left(\frac{\partial u}{\partial \varphi}\right) = \left(\frac{\partial}{\partial x}\cos\alpha + \frac{\partial}{\partial y}\sin\alpha\right)\left(\frac{\partial u}{\partial x}\cos\alpha + \frac{\partial u}{\partial y}\sin\alpha\right)$$

$$= \cos^2\alpha \frac{\partial^2 u}{\partial x^2} + \sin\alpha\cos\alpha \frac{\partial^2 u}{\partial y \partial x} + \cos\alpha\sin\alpha \frac{\partial^2 u}{\partial x \partial y} + \sin^2\alpha \frac{\partial^2 u}{\partial y^2}$$

$$= \cos^2\alpha \frac{\partial^2 u}{\partial x^2} + 2\sin\alpha\cos\alpha \frac{\partial^2 u}{\partial y \partial x} + \sin^2\alpha \frac{\partial^2 u}{\partial y^2} \quad \text{--- (3)}$$

$$\frac{\partial^2 u}{\partial \eta^2} = \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \eta}\right) = \left(-\sin\alpha\frac{\partial}{\partial x} + \cos\alpha\frac{\partial}{\partial y}\right)\left(-\sin\alpha\frac{\partial u}{\partial x} + \cos\alpha\frac{\partial u}{\partial y}\right)$$

$$= \sin^2\alpha \frac{\partial^2 u}{\partial x^2} - \sin\alpha\cos\alpha \frac{\partial^2 u}{\partial y \partial x} - \sin\alpha\cos\alpha \frac{\partial^2 u}{\partial x \partial y} + \cos^2\alpha \frac{\partial^2 u}{\partial y^2}$$

$$= \sin^2\alpha \frac{\partial^2 u}{\partial x^2} - 2\sin\alpha\cos\alpha \frac{\partial^2 u}{\partial y \partial x} + \cos^2\alpha \frac{\partial^2 u}{\partial y^2} \quad \text{--- (4)}$$

Adding eqn (3) and (4)

$$\frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial \eta^2} = \frac{\partial^2 u}{\partial x^2}(\cos^2\alpha + \sin^2\alpha) + \frac{\partial^2 u}{\partial y^2}(\sin^2\alpha + \cos^2\alpha)$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$ Proved Ans